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ABSTRACT

This is one in a series of SMSG supplementary and enrichment pamphlets for high school students. This series makes available expository articles which appeared in a variety of mathematical periodicals. Topics covered include: (1) Srinivasa Ramanujan; (2) Minkowski; (3) Stefan Banach; (4) Alfred North Whitehead; (5) Waclaw Sierpinski; and (6) J. von Neumann. (MP)

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Mathematics is such a vast and rapidly expanding field of study that there are inevitably many important and fascinating aspects of the subject which do not find a place in the curriculum simply because of lack of time, even though they are well within the grasp of secondary school students.

Some classes and many individual students, however, may find time to pursue mathematical topics of special interest to them. The School Mathematics Study Group is preparing pamphlets designed to make material for such study readily accessible. Some of the pamphlets deal with material found in the regular curriculum but in a more extended manner or from a novel point of view. Others deal with topics not usually found at all in the standard curriculum.

This particular series of pamphlets, the Reprint Series, makes available expository articles which appeared in a variety of mathematical periodicals. Even if the periodicals were available to all schools, there is convenience in having articles on one topic collected and reprinted as is done here.

This series was prepared for the Panel on Supplementary Publications by Professor William L. Schaaf. His judgment, background, bibliographic skills, and editorial efficiency were major factors in the design and successful completion of the pamphlets.

PREFACE

A notable characteristic of the development of mathematics in the twentieth century is the amazingly large number of able minds that have contributed to the proliferating achievements of these eventful years. The names of many persons readily spring to mind: P. Alexandroff, E. Artin, S. Banach, G. D. Birkoff, L. E. J. Brouwer, R. Courant, J. Douglas, S. Eilenberg, P. Erdös, W. Feller, A. Gelfond, K. Gödel, J. Hadamard, G. Hardy, D. Hilbert, E. Hille, A. Kolmogorov, H. Lebesque, S. Lefschetz, H. Poincaré, L. Pontrjagin, W. V. Quine, B. Russell, W. Sierpiński, H. Steinhaus, A. Tarski, B. L. Van der Waerden, I. M. Vinogradov, H. Weyl, N. Wiener: the list could readily be extended to many times its length.

On what basis, then, might one select a mere handful of names to be encompassed within the few pages of a slender pamphlet without doing an injustice, by implication, to literally scores of other equally illustrious mathematicians? For one thing, many of these distinguished mathematicians are still living. For another, only a few have become more or less well known to the general public either through books written by them or about them, as for example, Bertrand Russell and Norbert Wiener. All of which still leaves one in a quandary. Yet a choice had to be made. The six individuals selected represent marked contrasts in more ways than one. Three of them began their careers under adverse circumstances; the others enjoyed certain advantages. Most of them are all but unknown to the average layman. With some exceptions, their major interests in mathematics lay in fields that are widely apart. But let these thumbnail sketches speak for themselves.

—William L. Schaaf

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The School Mathematics Study Group takes this opportunity to express its gratitude to the authors of these articles for their generosity in allowing their material to be reprinted in this manner: Julius Sumner Miller, who at the time his article was originally published was associated with Dillard University, New Orleans; William Hammerschmidt, a contributor to *Scripta Mathematica;* I. Malkin, associated with the Foster Wheeler Corporation of New York City; Hugo Steinhaus, a distinguished Polish mathematician, author of *Mathematical Snapshots;* Matthew M. Fryde, of Columbia University and Yeshiva University; and lastly, Herman H. Goldstine, of the School of Mathematics, Institute for Advanced Study, Princeton, N.J., and Eugene Wigner of the Palmer Physical Laboratory, Princeton University, N.J.

The School Mathematics Study Group is also pleased to express its sincere appreciation to the several editors and publishers who have been kind enough to permit these articles to be reprinted, namely:

SCRIPTA MATHEMATICA

William W. Hammerschmidt, *"Alfred North Whitehead,"* vol. 14 (March 1948), pp. 17-23.

I. Malkin, *"A Sad Anniversary,"* vol. 24 (Spring, 1959), pp. 79-81.

Hugo Steinhaus, *"Stefan Banach, 1892–1945,"* vol. 26 (March 1963), pp. 93-100.

Matthew M. Fryde, *"Waclaw Sierpiński—Mathematician,"* vol. 27 (August 1964), pp. 105-111.

SCIENCE

Herman Goldstine and Eugene Wigner, *"Scientific Work of J. von Neumann,"* vol. 125 (April 12, 1957), pp. 683-684.

SCHOOL SCIENCE AND MATHEMATICS

Julius Sumner Miller, *"Srinivasa Ramanujan,"* vol. 51 (November 1951), pp. 637-645.

FOREWORD

It may be debatable that Ramanujan was one of the most extraordinary mathematicians of our time, as has been claimed. It would perhaps be more accurate to describe him as a particular kind of mathematician, by which we mean that his creative capacities and interests were limited to a particular field, to wit, the theory of numbers. He had no concern whatever for geometry; he was not at all interested in mathematical physics, or for that matter, any aspect of applied mathematics; he did not embrace the broad sweep of mathematics as did Gauss or Poincaré or Hilbert.

This, however, in no way belittles his remarkable achievement and contributions in his chosen field. The marvel is that despite a limited formal education and poor circumstances he rose to such heights as he he did in so short a lifetime. He was preeminently a mathematician's mathematician, even as was his friend and mentor, the late G. H. Hardy, of Cambridge University, himself a distinguished authority in the theory of numbers.

The interested reader will find it most stimulating to read the article on "Srinivasa Ramanujan" in James Newman's *The World of Mathematics*, volume 1, page 368 ff. And while he is about it, he will find G. H. Hardy's essay, *A Mathematician's Apology*, unforgettably charming.

Srinivasa Ramanujan

A BIOGRAPHICAL SKETCH

Julius Sumner Miller

"I remember once going to see him when he was lying ill at Putney. I had ridden in taxi cab No. 1729, and remarked that the number seemed to me rather a dull one, and that I hoped it was not an unfavorable omen. 'No,' he replied, 'it is a very interesting number; it is the smallest number expressible as a sum of two cubes in two different ways.' I asked him, naturally, whether he knew the answer to the corresponding problem for fourth powers; and he replied after a moment's thought, that he could see no obvious example, and thought that the first such number must be very large." Euler gave $158^4+59^4=133^4+134^4$ as an example.)

Thus does Professor G. H. Hardy report on an incident with Ramanujan.

Srinivasa Iyengar Ramanuja Iyengar, to give him his proper name, was one of the most remarkable mathematical geniuses of all time. This is especially true when the circumstances of his birth and life are intimately considered. It is difficult, indeed, for the student of the history of science to encounter a more astonishing biography. As Einstein so classically puts it: "Nature scatters her common wares with a lavish hand but the choice sort she produces but seldom." We will see, in this sketch, how closely this borders on a miracle.

Ramanujan, as he is commonly known, was born of a Brahmin family in the ever-so-present poverty that abounds in India. His ancestry, it appears, contributed nothing noteworthy to his great gifts. His mother was a woman of strong character. In accordance with custom her father prayed to the famous goddess Namagiri to bless her with children, and on December 22, 1887, this son was born. He started school at five. His early years showed no unusual signs of his special abilities although he was remarkably quiet and meditative, and he led his class. While in the "second form" he expressed a curiosity about the "highest truth in Mathematics." In the "third form" he was taught that any quantity divided by itself was unity whereupon he asked if zero divided by zero was also equal to unity! While in the "fourth form" he studied trigonometry and solved all the problems in the text without any aid whatsoever. In the "fifth form" he obtained unaided Euler's Theorems for the sine and cosine. When he found that these were already proved he hid his papers

in the roof of his house. While in the "sixth form" he borrowed Carr's *Synopsis of Pure Mathematics* and this book appears to have awakened his genius. He took great delight in verifying all the formulae therein and since he had no aid whatsoever, each solution was original research. He entertained his friends with recitation of formulae and theorems and demonstrated his remarkable memory by repeating values of *pi* and *e* to any number of decimal places. In every respect, however, he was utterly simple in his habits and unassuming.

He took up Geometry a little and by "squaring the circle" approximated the earth's circumference with an error of only a few feet. The scope of Geometry being limited in his judgment, he concerned himself with Algebra and obtained several new series. He said of himself that the goddess of Namakkal inspired him in his dreams and the remarkable fact is that on rising from bed he would at once write down results although he was not able to supply a rigorous proof. These results he collected in a notebook which he later showed to mathematicians.

In December 1903 (he was now only 16) he matriculated at the University of Madras and in 1904 won the "Junior Subrahmanyam Scholarship" at the Government College in Kumbakonam, this being awarded for proficiency in English and Mathematics. His absorption in mathematics, however, which was no less than devotion, led him to neglect his other work, and he failed to secure promotion. Indeed, he could be found engaged in some mathematical inquiry quite unmindful of what was happening in the class, whether it be English or History or anything else. For the next few years he pursued independent work in mathematics "jotting down his results in two good-sized notebooks." In the summer of 1909 he married. His greatest need was employment and this was a difficult matter for his family was poor, his college career a dismal failure, and he himself without influence.

In search of some means of livelihood he went, in 1910, to see one Mr. V. Ramaswami Aiyar, the founder of the Indian Mathematical Society. Aiyar, himself a mathematician of first order, found Ramanujan's notebooks remarkable and knew at once that this man possessed wonderful gifts. Accordingly, it was arranged that this unusual genius meet one Ramachandra Rao, a "true lover of mathematics," and a man in position to assist him. In December of 1910 Rao interviewed Ramanujan and Rao's own report of this meeting is a classic:

"... a nephew of mine perfectly innocent of mathematical knowledge said to me, 'Uncle, I have a visitor who talks of mathematics; I do not understand him; can you see if there is anything in his talk?' And in the plentitude of my mathematical wisdom I condescended to permit Ramanujan to walk into my presence. A short uncouth figure, stout, unshaved, not overclean, with one conspicuous feature—shining eyes—walked in with a frayed notebook under his arm. He was

miserably poor. He had run away from Kumbakonam to get leisure in Madras to pursue his studies. He never craved for any distinction. He wanted leisure; in other words, that simple food should be provided for him without exertion on his part and that he should be allowed to dream on.

"He opened his book and began to explain some of his discoveries. I saw quite at once that there was something out of the way; but my knowledge did not permit me to judge whether he talked sense or nonsense. Suspending judgment, I asked him to come over again, and he did. And then he had gauged my ignorance and showed me some of his simpler results. These transcended existing books and I had no doubt that he was a remarkable man. Then, step by step, he led me to elliptic integrals and hypergeometric series and at last his theory of divergent series not yet announced to the world converted me. I asked him what he wanted. He said he wanted a pittance to live on so that he might pursue his researches."

With all speed Rao sent Ramanujan back to Madras saying that it was cruel to make an intellectual giant like him rot away in obscurity, and he undertook to pay his expenses for a time. Since he was not happy being a burden to anybody for long, he took a small job in the Madras Port Trust office.

In the meanwhile his mathematical work was not slackened and during these years (1911–1912) he made his first contributions to the Journal of the Indian Mathematical Society. His first long article was on "Some properties of Bernoulli's Numbers" and he contributed also a number of questions for solution. Mr. P. V. Seshu Aiyar, through whom Ramanujan's papers were communicated, describes the work thus:

"Ramanujan's methods were so terse and novel and his presentation was so lacking in clearness and precision, that the ordinary reader, unaccustomed to such intellectual gymnastics, could hardly follow him."

Hav'ng thus gained a little recognition he fell heir to every encouragement and was thus brought to communicate with Professor G. H. Hardy, then Fellow of Trinity College, Cambridge. His first letter to Hardy bears more eloquence than is reasonable to expect of him and, indeed, Hardy did not believe that his letters were entirely his own. In this connection Hardy said: "I do not believe that his letters were entirely his own. His knowledge of English, at that stage of his life, could scarcely have been sufficient, and there is an occasional phrase which is hardly characteristic. Indeed I seem to remember his telling me that his friends had given him some assistance. However, it was the mathematics that mattered, and that was very emphatically his." His first communication to Hardy is a classic worth reciting:

Madras, 16th January, 1913

"Dear Sir,

"I beg to introduce myself to you as a clerk in the Accounts Department of the Port Trust Office at Madras on a salary of only 120 per annum. I am now about 23 years of age. I have had no University education but I have undergone the ordinary school course. After leaving school I have been employing the spare time

at my disposal to work at Mathematics. I have not trodden through the conventional regular course which is followed in a University course, but I am striking out a new path for myself. I have made a special investigation of divergent series in general and the results I get are termed by the local mathematicians as 'startling.'"

(He proceeds to discuss his interpretations of a certain integral and says)

"My friends who have gone through the regular course of University education tell me that the integral is true only when n is positive. They say that this integral relation is not true when n is negative. ... I have given meaning to these integrals and under the conditions I state the integral is true for all values of n negative and fractional. My whole investigations are based upon this and I have been developing this to a remarkable extent so much so that the local mathematicians are not able to understand me in my higher flights.

"Very recently I came across a tract published by you styled *Orders of Infinity* in page 36 of which I find a statement that no definite expression has been as yet found for the number of prime numbers less than any given number. I have found an expression which very nearly approximates to the real result, the error being negligible. I would request you to go through the enclosed papers. Being poor, if you are convinced that there is anything of value I would like to have my theorems published. I have not given the actual investigations nor the expressions that I get but I have indicated the lines on which I proceed. Being inexperienced I would very highly value any advice you give me. Requesting to be excused for the trouble I give you.

I remain, Dear Sir, Yours truly,
S. Ramanujan"

The papers he enclosed contained a hundred or more mathematical theorems. Some of his proofs were invalid for, after all, he was ignorant of very much of modern mathematics. He knew little or nothing of the theory of functions of a complex variable. He disregarded the precepts of the Analytic Theory of Numbers. His Indian work on primes was definitely wrong. But Hardy puts it beautifully:

"And yet I am not sure that, in some ways, his failure was not more wonderful than any of his triumphs." (He had none of the modern weapons at his command.) "He had never seen a French or German book; his knowledge even of English was insufficient to enable him to qualify for a degree. It is sufficiently marvelous that he should have even dreamt of problems such as these, problems which it has taken the finest mathematicians in Europe a hundred years to solve, and of which the solution is incomplete to the present day."

In his second letter to Hardy he wrote as follows:

"... I have found a friend in you who views my labours sympathetically. This is already some encouragement to me to proceed." (Follows a comment on an infinite series.) "If I tell you this you will at once point out to me the lunatic asylum as my goal. ... What I tell you is this. Verify the results I give and if they agree with your results ... you should at least grant that there may be some truths in my fundamental basis. ...

"To preserve my brains I want food and this is now my first consideration. Any sympathetic letter from you will be helpful to me here to get a scholarship either from the University or from Government. ..."

Hardy sensed at once that here was a mathematician of the very highest class and he proceeded at once to arrange for Ramanujan to come to England. The Indian's caste prejudices, however, were very strong, and he declined to go. This was a heavy disappointment to Hardy. After many persuasive letters and the influence of Indian friends Ramanujan had almost made up his mind to go but a new difficulty arose. His mother would not consent. This was overcome by a most unusual episode. His mother suddenly announced that she had a dream in which she saw her son seated in a big hall amidst a group of important Europeans, and that the goddess Namagiri, who had blessed her with this son, had commanded her not to stand in his way. At this time another Fellow of Trinity, Mr. E. H. Neville, who was delivering a course of lectures at Madras and who was acting as an ambassador for Hardy in urging Ramanujan to come to England, wrote a note to the authorities at the University of Madras:

"The discovery of the genius of S. Ramanujan of Madras promises to be the most interesting event of our time in the mathematical world.... The importance of securing to Ramanujan a training in the refinements of modern methods and a contact with men who know what ranges of ideas have been explored and what have not cannot be overestimated....

"I see no reason to doubt that Ramanujan himself will respond fully to the stimulus which contact with western mathematics of the highest class will afford him. In that case his name will become one of the greatest in the history of mathematics and the University and the City of Madras will be proud to have assisted in his passage from obscurity to fame."

The University authorities at Madras approved a grant, good in England for two years, with passage and a sum for outfitting him. Of this Ramanujan arranged to have a certain portion allotted to the support of his family in India. He sailed for England on March 17, 1914, being then 27 years of age. In April he was admitted to Trinity College where his grant was supplemented. For the first time in his life he was free of anxiety and certain of food, clothing and lodging. His requirements were so painfully simple that out of the scholarship monies he was able to save a goodly bit. Indeed, Professor Hardy describes his tastes as "ludicrously simple."

Now with Ramanujan among them the British mathematicians found themselves in a very certain dilemma. His knowledge of modern mathematics, of what had been explored in the mathematical world, was limited beyond belief. Indeed, "The limitations of his knowledge were as startling as its profundity. Here was a man who could work out modular equations, and theorems of complex multiplication, to orders unheard of, whose mastery of continued fractions was, on the formal side at any rate, beyond that of any mathematician in the world, who had

found for himself the functional equation of the Zeta-function, and the dominant terms of many of the most famous problems in the analytic theory of numbers; and he had never heard of a doubly periodic function or of Cauchy's theorem, and had indeed but the vaguest idea of what a function of a complex variable was. His ideas as to what constituted a mathematical proof were of the most shadowy description. All his results, new or old, right or wrong, had been arrived at by a process of mingled argument, intuition, and induction, of which he was entirely unable to give any coherent account.

"It was impossible to ask such a man to submit to systematic instruction, to try to learn mathematics from the beginning once more. I was afraid, too, that if I insisted unduly on matters which Ramanujan found irksome, I might destroy his confidence or break the spell of his inspiration. On the other hand there were things of which it was impossible that he should remain in ignorance. . . . So I had to try to teach him, and in a measure I succeeded, though obviously I learnt from him much more than he learnt from me. In a few years' time he had a very tolerable knowledge of the theory of functions and the analytic theory of numbers. He was never a mathematician of the modern school, and it was hardly desirable that he should become one; but he knew when he had proved a theorem and when he had not. And his flow of original ideas shewed no symptom of abatement."

Everything went well until the spring of 1917. Ramanujan published his papers in the English and European journals. All reports of him carried the highest praise. Mr. Hardy's report is particularly fine:

> "Ramanujan has been much handicapped by the war. Mr. Littlewood, who would naturally have shared his teaching with me, has been away, and one teacher is not enough for so fertile a pupil. . . . He is beyond question the best Indian mathematician of modern times. . . . He will always be rather eccentric in his choice of subjects and methods of dealing with them. . . . But of his extraordinary gifts there can be no question; in some ways he is the most remarkable mathematician I have ever known."

In the spring of 1917 Ramanujan showed signs of being not well. Since return to India was out of the question, sea travel was dangerous and medical men in India were scarce, he was placed in a sanatorium. From this time on he was never out of bed for any length of time again. He recovered slightly and two years later sailed for home. The climate of England did not lend itself to his stamina in the first place nor was it conducive to his recovery after he showed positive evidence of being tubercular.

While in England he was elected a Fellow of the Royal Society, thus being the first Indian on whom the honor was conferred. His age was 30. This together with the fact that he was elected the very first time his

name was proposed is a tribute of first magnitude to his remarkable genius. This honor appears to have incited him to further production for it was during these days that he discovered some of his most beautiful theorems. In that same year he was elected a Fellow of Trinity College, a prize fellowship worth some 250 pounds a year for six years, with no duties and no conditions. At this time Hardy wrote to the University of Madras as follows:

"He will return to India with a scientific standing and reputation such as no Indian has enjoyed before, and I am confident that India will regard him as the treasure he is."

In honor of Ramanujan's contributions to Mathematics the University of Madras made a further grant of 250 pounds a year for five years as well as covering travel between England and India as he chose. But Ramanujan was a sick man and of his election to these honors Hardy said "and each of these famous societies may well congratulate themselves that they recognized his claims before it was too late."

Having returned to India he was put in the best medical care and many persons contributed to his support. One Indian gentleman covered his entire expenses, another gave him a house free. But these served him not too well. On April 26, 1920, just about a year after his return to India, he died. He left no children. Only his parents and his wife survived him.

Regarding his appearance and personality, "before his illness he was inclined to stoutness; he was of moderate height (5 feet 5 inches); and had a big head with a large forehead and long wavy dark hair. His most remarkable feature was his sharp and bright dark eyes. . . . On his return from England he was very thin and emaciated and had grown very pale. He looked as if racked with pain. But his intellect was undimmed, and till about four days before he died he was engaged in work. All his work on 'mock theta functions,' of which only rough indications survive, was done on his deathbed."

Ramanujan had very definite religious views and adhered even in England with unusual severity to the religious observances of his caste. He believed in the existence of a Supreme Being and possessed settled convictions about life and life-hereafter. "Even the certain approach of death did not unsettle his faculties or spirits . . . " Hardy relates: ". . . and I remember well his telling me (much to my surprise) that all religions seemed to him more or less equally true."

In his manner he was utterly simple and without conceit. Of these characteristics Hardy wrote: "His natural simplicity has never been affected in the least by his success." That he possessed a charitable heart is beautifully illustrated in this letter:

To The Registrar,
University of Madras.

Sir,

I feel, however, that, after my return to India, which I expect to happen as soon as arrangements can be made, the total amount of money to which I shall be entitled will be much more than I shall require. I should hope that, after my expenses in England have been paid, L50 a year will be paid to my parents and that the surplus, after my necessary expenses are met, should be used for some educational purpose, such in particular as the reduction of school-fees for poor boys and orphans and provisions of books in schools. No doubt, it will be possible to make an arrangement about this after my return.

I feel very sorry that, as I have not been well, I have not been able to do so much mathematics during the last two years as before. I hope that I shall soon be able to do more and will certainly do my best to deserve the help that has been given me.

I beg to remain, Sir,
Your most obedient servant,
S. Ramanujan

It remains finally to give some estimate of Ramanujan's mathematics. For this the only sensible and proper source is Professor Hardy's account for all of Ramanujan's manuscripts passed through his hands. Hardy edited all of them and rewrote the earlier ones completely. In some he collaborated. In this connection Hardy writes: "Ramanujan was almost absurdly scrupulous in his desire to acknowledge the slightest help." It is obviously impossible to give a value in the proper sense to Ramanujan's mathematics: all we can do is quote freely from Hardy's own remarks.

"... Some of it is very intimately connected with my own, and my verdict could not be impartial: there is much too that I am hardly competent to judge....

"... But there is much that is new, and in particular a very striking series of algebraic approximations to π. I may mention only the formula

$$\pi = \frac{63}{25}\,\frac{17+15\sqrt{5}}{7+15\sqrt{5}}, \qquad \frac{1}{2\pi\sqrt{2}} = \frac{1103}{99^2}$$

correct to 9 and 8 places of decimals respectively."

"... but the elementary analysis of 'highly composite' numbers — numbers which have more divisors than any preceding number — is most remarkable, and shews very clearly Ramanujan's extraordinary mastery over the algebra of inequalities...."

"... They contain, in particular, very original and important contributions to the theory of the representation of numbers by sums of squares."

"... But I am inclined to think that it was the theory of partitions, and the allied parts of the theories of elliptic functions and continued fractions, that Ramanujan shews at his very best."

"It would be difficult to find more beautiful formulae than the 'Rogers-Ramanujan' identities...."

"He had, of course, an extraordinary memory. He could remember the idiosyncracies of numbers in an almost uncanny way. It was Mr. Littlewood (I believe) who remarked that 'every positive integer was one of his personal friends.' "

"His memory, and his powers of calculation, were very unusual."

"It was his insight into algebraical formulae, transformations of infinite series, and so forth, that was most amazing. On this side most certainly I have never met his equal, and I can compare him only with Euler or Jacobi. He worked, far more than the majority of modern mathematicians, by induction from numerical examples."

"But with his memory, his patience, and his power of calculation, he combined a power of generalization, a feeling for form, and a capacity for rapid modification of his hypotheses, that were often really startling, and made him, in his own peculiar field, without a rival in his day."

"Opinions may differ as to the importance of Ramanujan's work, the kind of standard by which it should be judged, and the influence which it is likely to have on the mathematics of the future. It has not the simplicity and the inevitableness of the very greatest work; it would be greater if it were less strange. One gift it has which no one can deny, profound and invincible originality. He would probably have been a greater mathematician if he had been caught and tamed a little in his youth; he would have discovered more that was new, and that, no doubt, of greater importance. On the other hand he would have been less of a Ramanujan, and more of a European professor, and the loss might have been greater than the gain."

NOTE: This sketch borrows with obvious freedom from *Collected Papers of Srinivasa Ramanujan*, Edited by G. H. Hardy, P. V. Seshu Aiyar, and B. M. Wilson. Cambridge University Press 1927. Any attempt to paraphrase the eloquence of their recitation would border on ridiculous vanity! JSM

FOREWORD

Most great contributions in mathematics and science are built upon the achievements of one's predecessors. This is eminently true of Einstein's theory of relativity. Had it not been for the development of Minkowski's four-dimensional geometry, the mathematical basis for relativity theory, would at least in part, have been impossible.

Minkowski is also to be credited with equally significant contributions to the theory of numbers. He was especially ingenious in interpreting arithmetical problems geometrically, a procedure which often afforded unexpected insight into analytical processes.

Today the geometry of numbers, created virtually by Minkowski alone, has expanded into a highly significant branch of mathematics. As in the case of many profound and difficult fields of mathematics, it is not always easy to convey the essence of a particular discipline in brief. Number theory is no exception. One of the basic principles, which, according to E. T. Bell, is so simple and obvious as to appear ridiculous, is the following: if $(n + 1)$ things are stored in n boxes, and no box is empty, then exactly one of the boxes must contain two things. This principle of geometrized arithmetic can be applied to a tricky puzzle perpetrated by some wag: What are the necessary and sufficient conditions that there shall be at least two human beings in the world with the same number of hairs on their heads?

A Sad Anniversary

I. Malkin

January 12 of 1909 was a tragic day in the history of Mathematical Sciences: on that day Hermann Minkowski, one of the great mathematicians of his time, died at the age of 44 at the peak of a brilliant scientific career.

For those who know little of this great man of science, the following short biographical sketch might be of interest [an extensive biography of Hermann Minkowski will be found in the first volume of his "Gesammelte Abhandlungen" edited by his close friend and colleague, the celebrated David Hilbert (1862–1943) of the Department of Mathematics, University of Goettingen, and published by B. G. Teubner in Leipzig, Germany].

Hermann Minkowski was born on June 22, 1864, in a Jewish family in Russia [Hilbert mentions "Alexoten (perhaps Alexotin? I. M.) in Russia" as the birthplace, but this writer must confess that a place of this name is not known to him]. He was brought to Germany as a child and there at the age of 8 he entered the Altstaedtisches Gymnasium (high school) of Koenigsberg in Prussia. His extraordinary abilities made it necessary for the teachers to transfer him from one class to another at an accelerated tempo, and so it happened that at the age of 15 he already obtained his graduation certificate, whereupon he immediately started his studies at the University of Koenigsberg. He graduated from the University of Berlin with a Ph.D. at the age of 19.

In April 1881 the Academy of Sciences of Paris posed the problem of resolving integer numbers into a sum of five squares. On May 30, 1882, Minkowski submitted a solution to the Academy, which was a most startling achievement. It went far beyond the scope of the problem and constituted an outstanding contribution to the theory of numbers. No wonder that it produced a veritable sensation at the Academy. The eighteen-year-old scientist was awarded the "Grand Prix des Sciences Mathématiques." Some of the Paris newspapers were protesting vigorously against the decision of the Academy, because Minkowski had submitted his work in German, which was in contradiction with the

regulations of the Academy. The protest was, however, fruitless because the contribution was found to be too important to permit paying attention to the question of language. Such great men of French mathematics as Camille Jordan (1838–1922), Joseph Bertrand (1822–1900), and particularly Charles Hermite (1822–1901) were supporting Minkowski with all the weight of their authority. "Travaillez, je vous prie, à devenir un géomètre éminent" ("Please, keep working, in order to become an eminent mathematician"), Camille Jordan wrote to him. Highly interesting is a letter of Charles Hermite to the young genius: ". . . At the first glance I recognized that you went far beyond my investigations and that you have discovered for us entirely new ways. . . . I am full of astonishment and joy at your results and methods. You are so kind as to call my old research works a point of departure for your magnificent contribution, but you have left them so far behind yourself that they cannot claim now any other merits but to have suggested to you the direction in which you have chosen to proceed."

The mathematical interests of Minkowski were not confined to a particular narrow field; during the years 1884–1892 he was working, together with his colleagues Adolf Hurwitz (1859–1919) and David Hilbert, very intensively in a great variety of mathematical sciences, including mathematical physics. The great man of experimental physics, Heinrich Hertz (1857–1894), the discoverer of the identity of the electromagnetic and the light waves, took him as his assistant to the University of Bonn, and later Minkowski once stated that perhaps he would have become a physicist, if Hertz would have lived longer. Some of Minkowski's publications deal with problems of mathematical physics, as we presently shall see.

An important later period of his scientific, especially pedagogic activities is closely linked with his work at the Institute of Technology of Zurich, Switzerland. In connection with some contributions to the Encyclopaedia of Mathematical Sciences his classes in pure mathematics were amplified sometimes by lectures in special domains of mathematical physics. These latter lectures won him the admiration of a young student by the name of Albert Einstein.

Minkowski is popularly known for his contributions to three fields: (1) theory of numbers, as already implied in the above account of his most outstanding success of 1882–83 in Paris; (2) geometry, as the interested reader will convince himself by a study of the well-known textbooks of Wilhelm Blaschke, published by Julius Springer in Berlin, Germany; (3) theory of relativity, to which he completed an investigation of fundamental importance shortly before his death. He dealt with the subject in his last public appearance, which took place on Sep-

tember 21, 1908, in Koelin (Cologne), Germany. Students of relativity know well the monograph entitled "Das Relativitaetsprinzip" by H. A. Lorentz (1853–1928), A. Einstein and H. Minkowski. The importance of Minkowski's contribution to the theory of relativity is characterized by Einstein in the following statement: "Ohne den wichtigen Gedanken Minkowskis waere die allgemeine Relativitaets-theory vielleicht in den Wideln stecken geblieben" (Without the important ideas of Minkowski the generalized theory of relativity perhaps would still be in its infancy").

At the time of his death Minkowski was Professor at the famous Department of Mathematics of the University of Goettingen.

We conclude this short biographical sketch by reporting here a detail which is very typical of the purely human aspects of Minkowski's character. Already during the first semester at the University he was awarded a money prize for the solution of a prize problem in mathematics. He withdrew his claims to the money in favor of a needy colleague, and his relatives never heard anything about his first academic success.

His thoughts and ideas were devoted to science almost to his very end. His death, caused by appendicitis, could not be averted by a belated operation.

Hermann Minkowski deserves our deep admiration as a great and kind-hearted man of science. His name will be honored forever in the history of mathematics.

LITERATURE USED

1. *Helle Zeit-Dunkle Zeit*, in memoriam Albert Einstein, a collection of articles edited by Carl Seelig and published by Europa Verlag, Zurich, Stuttgart, Wien; printed in Zurich, Switzerland, 1956.

2. *Hermann Minkowski, Gesammelte Abhandlungen*, edited by David Hilbert, published by B. G. Teubner, Leipzig, Germany, in 1911.

3. *Philipp Frank, Einstein*, His Life and Times, Alfred Knopf, New York, 1947. Remark: It must be admitted that there is a certain contradiction between p. 20 of [3] and p. 21 of [1] (concerning Einstein's opinion about the classes held by Minkowski at the Institute of Technology of Zurich).

FOREWORD

To set forth in simple language the significance of Stefan Banach's contributions to the highly technical field of mathematical analysis is frankly an almost impossible task. Yet his compatriot Hugo Steinhaus has succeeded rather admirably in conveying to the non-mathematician the spirit and general nature of Banach's work. Suffice it to say here that in a sense his contributions came as a climax, or at least a high point, in the long struggle toward greater abstraction and generalization in mathematics which began when Gauss created his hypercomplex integers about 1830. The processes of "arithmetization of algebra" and "geometrizing arithmetic" have been going on ever since in the mathematician's relentless pursuit in search of "structure."

Stefan Banach,[1] 1892-1945

Hugo Steinhaus

Stefan Banach was born on March 20, 1892, in Cracow. His father, a clerk in the Cracow regional railways board, was called Greczek and came from Jordanów, from a highlander family. The history of Banach's childhood years is not known in any detail; but shortly after his birth he was, to be brought up, given to a washerwoman whose name was Banachowa, and who lived in an attic in Grodzka Street (No. 70 or 71); he never again met his mother, so that in fact he hardly knew her. His father did not care for the boy either, and by the time he was fifteen Banach had to make his own living by giving lessons. His favourite teaching involved mathematics, which he studied on his own and even at secondary school, reading the French book by Tannery on the theory of real functions; how he acquired his knowledge of French, is not known.

Before the First World War, he used to attend lectures under Stanislaw Zaremba at the Cracow University, but never in any regular manner and then for a short time only, after which he moved to the Lwów Institute of Technology. There, he passed his so-called "first examination," taken after the first two years of engineering studies. Upon the outbreak of the war in 1914, he returned to Cracow.

On a walk along the Cracow Green Belt one summer evening in 1916, I overheard a conversation, or rather only a few words; but these, "the Lebesgue integral," were so unexpected that I went up to the bench and introduced myself to the speakers—Stefan Banach and Otto Nikodym discussing mathematics. They told me they had a third member of their little group, Wilkosz. The three companions were linked not only by mathematics, but also by the hopeless plight of young people in what was then the fortress of Cracow—an insecure future, no opportunities for work and no contacts with scientists, foreign or even Polish. This indeed was the atmosphere in the Cracow of 1916. Still, it did not hamper the three friends from sitting long hours in a café and solving their problems amidst the crowd and the noise; in any case, noise was no obstacle to Banach, who for some unknown reason actually preferred a table close to the orchestra.

[1] Paper read on September 4, 1960, at the conference held in Warsaw on the occasion of the 15th anniversary of the death of Stefan Banach, and organized by the Polish Academy of Sciences Mathematical Institute.

Reprint, *Rev. Polish Acad. Sci.*, Vol V, 1960.

Banach's aspiration then was to become an assistant of mathematics in the Lwów Technical University. This was fulfilled in 1920, when Antoni Łomnicki appointed him to the post. By that time Banach was author of a paper on the mean convergence of sums of Fourier's partial series. It was a problem I had posed to him in 1916, at that first meeting in the Cracow Green Belt. I had myself for long been trying to solve it and it was something of a surprise to me when Banach arrived at a negative answer. This he communicated to me several days later, with the reservation that he did not know the example of Du Bois-Reymond. Our common note on the subject was, after a long delay, at last presented by S. Zaremba to the Cracow Academy of Knowledge in 1918.

On his arrival in Lwów, Banach's situation changed radically. His material problems were now solved. He married and started to live at the University house in St. Nicolas Street. In 1922, his doctor's thesis, "Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales," was published in Volume III of *Fundamenta Mathematicae* (pp. 133–181).

This, his seventh paper, was the first devoted to the theory of linear operations. In the same year he received his *venia legendi*. The usual university procedure was not applied to his case: he had been awarded a doctorate even though he had never in fact graduated, and was appointed professor immediately upon acceptance of his *venia legendi*. He was then thirty. There was no lack of recognition also from other quarters. In 1924, he was admitted to corresponding membership in the Polish Academy of Sciences and Arts and in 1930 was awarded the prize of the City of Lwów, which was followed in 1939 by the Grand Prize of the Academy. Still, it is hard to understand today how it was that the Academy never offered a full membership to this child of the Cracow streets. But, the Lwów mathematicians at once realized that Banach was destined to raise Polish mathematics to international fame. Until his arrival in the city, there had been no Lwów mathematical school to speak of, since Waclaw Sierpinski had soon after the First World War returned to Warsaw, which he had had to leave while it still lasted, and Zygmunt Janiszewski had died soon afterwards.

The speciality of the Lwów school of the period between the wars can be best defined as the theory of operations. Banach set to work on linear functionals, such as integrals. He showed that the concept of integral could be so extended as to embrace all functions, while at the same time retaining the properties postulated by Lebesgue: even though this concept is ineffective, the proof and its conduct (*Fundamenta Mathematicae*, 1923) give evidence of Banach's power. His *magnum opus* is the book on linear operations. Published in 1932 as the first volume of *Mono-*

grafie Matematyczne (Mathematical Monographs, Warsaw, VII, pp. 254), it is now known to the entire mathematical world under the title *Théorie des opérations linéaires*. Its success lies in that, by way of what is known as Banach spaces, one can now solve in a general manner numerous problems which before required separate treatment and no little ingenuity. There had been other mathematicians, great and less great, who had attempted before Banach to construct a theory of operations. I recall hearing the eminent Göttingen scholar Edmund Landau say about the book *Operazioni distributive* by Pincherle: "Pincherle has written a book in which he has not proved any single theorem," and this was in fact the case. But there were more powerful competitors also. Let us quote from what Norbert Wiener, the creator of cybernetics, writes in his autobiography, *I am a Mathematician*, published in London in 1956. In it he mentions Fréchet, who had been the first to give the shape of the linear functional in space L^2, but who had not attempted a a set of postulates which would define such a general space so as to make of L^2 merely one of many examples. This Wiener ascribes to himself. He relates how Fréchet, whose guest he was in 1920 on the occasion of the Strasbourg mathematical congress, showed to him in "some Polish mathematical journal" an article by Banach. Fréchet was "quite excited" by the fact that Banach had arrived a few months earlier than Wiener at a set of axioms for an infinite-dimensional vector space, identical with that of Wiener's. "Thus the two pieces of work," Wiener says, "Banach's and my own, came for a time to be known as the theory of Banach-Wiener space." "For a short while," he continues, "I kept publishing a paper or two on this topic, but I gradually left the field. At present, these spaces are quite properly named after Banach alone." After this confession Wiener devotes a few pages of his autobiography to this collision and explains why he "left the field"; it seemed to him that Banach's theory involved "rather thin and formal work," which could not have been endowed with "a sufficiently unobvious body of theorems." He now admits he was wrong, since for the 34 years, which have passed since the Strasbourg congress, Banach's theory "has remained a popular direction of work" and "only now is it beginning to develop its full effectiveness as a scientific method."

Banach's fame had reached the United States already in 1934 when J. D. Tamarkin had published in the *Bulletin of the American Mathematical Society* (vol. 40, pp. 13–16) some pages about the "Théorie des opérations linéaires." He writes: "The present book [. . .] represents a noteworthy climax of long series of researches started by Volterra, Fredholm, Hilbert, Hadamard, Fréchet, F. Riesz, and successfully continued by Steinhaus, Banach and their pupils." He continues: "The theory of

linear operations is a fascinating field in itself but its importance is still more emphasized by numerous beautiful applications to various problems of analysis and function theory. . . ."

One of the most gifted among Banach's young friends, Stanislaw Ulam, writes in the same *Bulletin* (vol. 52, July 1946, pp. 600–603): "News came recently that Stefan Banach died in Europe shortly after the end of the war. The great interest aroused in this country by his work is well known. In fact, in one of Banach's main fields of work, the theory of linear spaces of infinitely many dimensions, the American School has developed and continues to contribute very important results. It was a rather amazing coincidence of scientific intuition which focused the work of many mathematicians, Polish and American, on this same field . . ." "Banach's work brought for the first time in the general case the success of the methods of geometrical and algebraic approach to problems in linear analysis—far beyond the more formal discoveries of Volterra, Hadamard, and their successors. His results embrace more general spaces than the work of such mathematicians as Hilbert, F. Riesz, von Neumann, Stone, and others. Many mathematicians, especially the younger ones of the United States, took up this idea of geometrical and algebraic study of linear function spaces, and this work is still going on vigorously and producing rather important results."

Surely, those judgments by prominent scientists (one of whom played a leading role in calculating the thermonuclear hydrogen reaction) will be sufficient as proof of Banach's achievement of a major place in the history of the development of a highly important and novel field of analysis and in pushing himself to the forefront of a group of outstanding mathematicians, who had earlier tested their strength in a similar direction.

Permit me, as a witness of Banach's work, to say that he had a clarity of thought which Kazimierz Bartel, professor of geometry and prime minister, once called "positively unpleasant." He had never hoped for happy coincidences or for fond wishes to be realized, and he liked to say that "hope is the mother of fools"; this contempt of foreknowledge he applied not only to mathematics, but also to political prophecies. He resembled Hilbert in that he attacked problems head-on, after excluding by means of examples all possible side roads, concentrating all his forces on the road that remained and which led straight to the goal. He believed that a logical analysis of a problem, conducted rather like a chess-player analyses a difficult position, must lead to proving or disproving a proposition.

Banach's importance is not restricted to what he achieved on his own in the theory of linear operations; the list of his 58 works includes papers

written jointly with other mathematicians — both those working in his own speciality and in others. To this category belongs the paper on the division of sets into congruent parts, written with Tarski (No. 13 in the list of works: *Fundamenta Mathematicae*, VI, 1924, pp. 244–277). This is a theme resembling the school method of proving the Pythagorean proposition by cutting a large square into parts, out of which two small squares can be assembled. In three dimensions the result is unexpected: a sphere can be broken up into several parts in such a manner that from them two spheres, each equal in size to the original, can be reconstructed. For my part, I was extremely impressed by a short paper of Banach's in *Proceedings of the London Mathematical Society* (vol. 21, pp. 95–97). The problem consists in finding an orthogonal set complete in L^2, but incomplete in L. Banach chooses here the function $f(t)$ integrable (L), $\int_0^1 f(t)dt = 1$, but such that $\int f^2(t)dt = \infty$, defines $\{\varphi_n(t)\}$ as the sequence of all trigonometric functions $\{\cos \pi nt, \sin \pi nt\}$, and defines the numerical sequence $\{c_n\}$ by the relation $\int_0^1 f(t)\varphi_n(t)dt = c_n$; if now the sequence $\{\psi_n(t)\}$ is defined by $\psi_n(t) = \varphi_n(t) - c_n$, the result will, of course, be $\int_0^1 f(t)\psi_n(t)dt = 0$ for all n.

When we now orthogonalize and normalize the sequence $\{\psi_n\}$, we shall get the required sequence $\{\gamma_n(t)\}$. The idea of the proof consists in that the auxiliary sequence $\{\varphi_n(t)\}$ has *not* the property which is demanded of the sequence sought.

Well known also are Banach's papers concerning the convergence of functionals, work on which had been commenced by one of Banach's colleagues, generalized by Banach himself and brought to their final shape by S. Saks (*Fundamenta Mathematicae*, No. IX, 1927, pp. 50–61). Banach was further interested in the problem of complanation, i.e., in a definition of the concept of the area of curved surfaces, and his very pertinent definition is still a subject of study [in, among other places, Lwów, by Professor Kovanko]. Unfortunately, nobody knows how to reproduce the basic lemma, which is essential for showing the agreement of Banach's definition with classical ones. It must be stated with regret that many precious results achieved by Banach and his school perished — a great loss to Polish science — in consequence of the lack of meticulousness shown by members of the school, and above all by Banach himself.

Of great beauty also is the idea of replacing the classical definition of the variation of the function $y = f(x)$ by one better corresponding to Lebesgue's age, namely by the integral $\int_{-\infty}^{+\infty} L(\eta)d\eta$, in which $L(\eta)$ denotes the number of intersections of the curve $y = f(x)$ by the straight line $y = \eta$; readers may perhaps be interested to know that this formulation is of practical significance, since, for instance, it allows the rapid calcu-

lation in "dollar days" of bank credits frozen in factory stocks in the shape of raw materials waiting to be processed.

I do not intend to pursue the numerous and important items on the list of works by the creator of the Lwów school and founder of *Studia Mathematica*, a journal which played a not insignificant role in the development of that school and in the history of the theory of linear operations. I prefer to come back to Banach's personality, to his direct influence on his milieu. Stefan Banach became a full professor in 1927, but neither before that date nor after was he ever a don in the pompous sense of the word. He was an excellent lecturer, never getting lost in detail and never covering the blackboard with a profusion of complicated signs. He did not care for the perfection of verbal form. All humanistic elegance was alien to him and all his life he retained certain features of the Cracow street urchin in both his behaviour and speech. The formulation of thoughts in writing was always a source of some difficulty to him and he would compose his manuscripts on loose sheets of paper torn from a notebook. When parts of the text had to be changed, he would cut out the now unnecessary passages and glue the rest on to a clean piece of paper, on which he would then write the new version. Were it not for the help of his friends and assistants, his earliest works would never have reached the printers. After all, he never wrote letters and never answered questions sent to him by letter. He did not go in for logical speculation, even though he understood such speculations perfectly. He was not attracted by the practical applications of mathematics, even though he could have certainly engaged upon them; he did, in fact, lecture on mechanics at the Institute of Technology. He used to say that mathematics can be identified by a specific kind of beauty and will never be reduced to a rigid deductive system, since sooner or later it breaks out of any formal framework and creates new principles. For him, the value of a mathematical theory was decisive, its value *per se*, not the utilitarian one. His foreign competitors, in the theory of linear operations, employed either too general spaces, in which case the results achieved could not but be trivial or made too many assumptions about those spaces, which narrowed down the scope of applications to few and artificial examples. Banach's genius was expressed in his finding of the golden mean. This ability to get to the heart of the matter shows Banach as a true mathematician. Wiener entitles his autobiography *I am a Mathematician*, Banach can claim the name of a mathematician *par excellence*.

He was able to work at any time and at any place. Not accustomed to comfort and with no need of it, his professorial salary should have really been quite sufficient for him. But his weakness for café life and total

lack of middle-class frugality and orderly approach to matters of every-day life loaded him with debts and finally landed him in a very difficult situation indeed. Trying to disentangle himself, he sat down to writing text-books. It was thus that his *Rachunek różniczkowy i całkowy* (Differential and Integral Calculus) in two volumes came to be written; the first of these was published by "Ossolineum" (1929, pp. 294), the other by Książnica-Atlas (1930, pp. 248). This text-book, compactly and clearly presented, enjoyed and still enjoys wide popularity among university students in their early years of study.

Most of Banach's time and powers were consumed in the writing of secondary school text-books of arithmetic, algebra and geometry. Some of them he wrote jointly with Sierpiński and Stożek, others on his own. None of them is a copy of an earlier school text-book. Thanks to his experiences as a private tutor, Banach was well aware of the fact that each definition, each argument and each exercise was a problem for a text-book author who really cared about its didactic value. To my mind, Banach lacked only one of the many talents necessary to an author of school text-books – the ability to see things in space. A fruit of the experiences gained in the course of the many lectures on mechanics delivered by invitation at the Lwów Institute of Technology was *Mechanika w zakresie szkół akademickich* (Mechanics – a University course, *Monografie Matematyczne* 8, 9). This two-volume work, first published in 1938, was reissued in 1947 and appeared a few years ago also in an English translation

To obtain some idea of Banach's contribution to science as a whole, and to Polish science first and foremost, the names of some of his students are important. Stanisław Mazur and Władysław Orlicz were his direct disciples, and it is they who are now Poland's representatives of the theory of operations, their names on the cover of *Studia Mathematica* being a symbol of a direct continuation of Banach's scientific programme, which had once found a vivid expression in this journal. Stanisław Ulam, whom I have already mentioned, and who owed to Kuratowski his mathematical initiation, has been also captured by Banach's attraction. Banach, Mazur, and Ulam: this used to be the most important table at the Scottish Café in Lwów. It was there that the meetings took place about which Ulam wrote in Banach's obituary that "it was hard to outlast or outdrink Banach during these sessions." One such session lasted seventeen hours, its result being the proof of an important theorem relating to Banach's space. Still, nobody noted it down and nobody would now be able to reconstruct it; probably, after the session was over, the café charwoman as usual washed the table top, covered with pencil marks. This was in fact what happened to many theorems proved by

Banach and his students. And so, it is greatly to the credit of Mrs. Lucja Banach, now buried in the Wroclaw cemetery, that she bought a thick notebook with hard covers and entrusted it to the headwaiter of the Scottish Café. In it, mathematical problems were now written down on one side of successive pages, so that possible answers could one day be put in one of the free pages, next to the texts of the questions. This unique "Scottish Book" was placed at the disposal of every mathematician who demanded it on coming to the café. Some propositions were included in it with the promise of a prize for solution, the prizes ranging from a small cup of coffee to a live goose. If any reader now smiles condescendingly at such methods of mathematical research, he should remember that, in accordance with Hilbert's view, the formulation of a problem is halfway to its solution; that a published list of unsolved problems compels a search for the answers and is a perpetual challenge to all who want to measure their powers against their intentions; such a state of intellectual emergency breeds a genuinely scientific atmosphere.

The most outstanding of those of Banach's students who fell victim to the uniformed and swastika-decorated murderers, was undoubtedly J. Schauder, winner of the international Metaxas prize, jointly with Léray. It was Schauder who first realized the importance of Banach spaces to the marginal problems of partial differential equations. The difficulty lay in the selection of proper norms; this was overcome by Schauder and it was thanks to this young mathematician that priority in such a classical theory as partial differential equations had to be shared by France with Poland.

A grim shadow on Banach's later years was cast by the events of the Second World War. In 1939–41 he served as Dean of the Mathematical and Natural Sciences Faculty in the Lwów University, and even became corresponding member of the Kiev Academy. After the German entry into Lwów in late June, 1941, however, he had to work as a louse-feeder in Professor Weigl's Bacteriological Institute. He spent several weeks in prison because persons engaged in smuggling German marks were found at his flat. Before the matter was cleared up, he had succeeded in proving in prison a certain new theorem.

Banach was above all a mathematician. He had little interest in political matters, even though he always had a clear view of the situation in which he happened to find himself. Nature did not impress him at all. The arts, literature, the theatre were but secondary diversions, which could fill out, and only seldom, the short pauses from work. He did appreciate, on the other hand, a company of friends over a drink. Thus, the concentration of his intellectual resources in a single direction was

wholly unimpeded. He was not prone to delude himself. and knew full well that only a few people in a hundred can understand mathematics. He told me once: "You know, old chap, what I'm going to tell you? The humanities are more important than mathematics in the secondary school; mathematics is too sharp an instrument. It's not for kids."

It would be wrong to imagine Banach as a dreamer, apostle or ascetic. He was a realist, who did not resemble even physically prospective saints. I do not know whether there still exists today — it certainly did exist even 25 years ago — the ideal of a Polish scientist created on the basis not so much of observation of real scientists as of the spiritual needs of the age, which were best expressed by Stefan Zeromski. A scientist of that kind was supposed to work far from worldly pleasures for some indefinite "society," with the ineffectuality of his work forgiven him in advance; it did not really matter that in other countries scientists were appraised not by the greatness of personal self-denial. but by the lasting things they had given science. The Polish intelligentsia remained even between the two wars under the spell of this ideal. Banach never surrendered to it. Healthy and strong, a realist bordering on the cynic, he has given to Polish science — and to Polish mathematics in particular — more than anyone else has given. None contributed more than he to eradicate the harmful view that in scientific emulation a lack of genius (or perhaps even a lack of talent) can be balanced by other virtues, which have the distinction of being hard to define. Banach was aware of both his own qualities and the qualities he was creating. He emphasized his highlander origin and had a somewhat superior attitude to the versatilely educated intellectual without portfolio.

He lived to see Germany defeated. but died soon after in Lwów on August 31, 1945. He was given a funeral at the Ukrainian Republic's expense. A street in Wroclaw[2] has been named after him and his collected works will be published by the Polish Academy of Sciences.

His greatest merit was to overcome. once and for all. and completely to destroy the complex consisting in the Poles' feeling of inferiority in the sciences. a complex masked by pushing to the fore of mediocrities. Banach never gave way to that complex. In his personality. the flash of genius was combined with some astounding inner imperative. which kept telling him. in Verlaine's words: "Il n'y a que la gloire ardente du métier." And mathematicians know well that to their *métier* there is the same mystery as to the craft of the poets.

[2] Recently Warsaw followed this example.

FOREWORD

Most men of achievement are content to have led one fruitful life. Alfred North Whitehead could claim three distinguished careers. For nearly a quarter of a century he was a lecturer in mathematics at Trinity College of Cambridge University in England. Then came a brief interlude of slightly more than a decade during which he was professor of applied mathematics at the University of London. This was followed by the third epoch of his life, which began at the age of 63, when in 1924, he became by invitation professor of philosophy at Harvard University in Cambridge, Mass. In many ways this was the most brilliant period of his life. Of all his books — some twenty or so — half of them were written while he was in America.

Whitehead's chief claim to fame rests upon his contributions to mathematics and logic, to the philosophy of science, and to the study of metaphysics. The monumental three-volume work *Principia Mathematica*, written in 1910–1913 in collaboration with Bertrand Russell, by itself is enough to assure Whitehead a permanent place in the hall of fame of mathematics. Its impact upon the foundations of mathematics and upon metamathematics is still felt today, more than half a century later. Among his most significant philosophical works we must include *Process and Reality*, *Adventures of Ideas*, and *Modes of Thought*.

If it were possible to capture the spirit of this gentle, serene benign figure in a few words, perhaps the following closing sentence of a letter to President Conant of Harvard would serve as well as any other:

"During my life I have had the great happiness of teaching in two countries which have contributed so greatly to learning and to the dignity of mankind."

Alfred North Whitehead

William W. Hammerschmidt

One of the most interesting men of our modern intellectual world is Alfred North Whitehead. His interest is due not only to the range of his learning and speculation—the emulation of which would have reduced most men to triviality—but also to the creative and imaginative power which he applied to his learning. So, for example, when he was asked to write the articles for the Encyclopedia Britannica on Non-Euclidian Geometry and the Axioms of Geometry, he was already one of the world's foremost mathematical logicians, and when he was over sixty he was asked to become Professor of Philosophy at Harvard University, although he had never before held an academic post outside of mathematics. Of course his creative ability can become apparent only by reading his works.

Whitehead was born into a mid-Victorian culture eighty-seven years ago, on the fifteenth of February, 1861. From early childhood he was closely associated with educational life. Both his father and his grandfather were successful schoolmasters, his grandfather having become head of a private school in Ramsgate on the Isle of Thanet, Kent, at the age of twenty-one, and his father becoming head of the same school at the age of twenty-five. And in 1880, when Whitehead was entering Cambridge, his two older brothers were in academic life, one as a tutor at Oxford, and the other as a schoolmaster. Not unnaturally, Whitehead's interest centered in the educational world also.

His education was typically mid-Victorian. At fourteen he was sent to school as Sherborne in Dorsetshire. There, in surroundings rich in the relics of history and tradition, he studied the normal subjects of the English schools—Latin, Greek, the historians (Thucydides, Herodotus, Xenophon, Sallust, Livy, Tacitus), some French, some science, and more than the usual amount of mathematics, in which already he had a strong interest. Whitehead says that this classical education had a direct relevance to his own times for the English schoolboy of that age. The historians were read in the light of English politics and history and statesmanship. The navy of Athens was comparable to the British navy, ruling the seas, and catching the enemy at anchor in bays or rounding capes. Athens was the ideal city, and the Athenian ideal of the golden mean

(seldom realized by her politicians and generals) entered into the philosophy of English statesmanship. To these English boys, history was not a dead subject, but a link with a still living past. And the past had left its visible mementos around and at the school in Dorsetshire—as well as in Kent where Whitehead spent his boyhood—relics of the Romans, the Normans, and the Saxons. It was from such a background of study and history that Whitehead derived his later emphasis on the importance in education of implanting in students "a right conception of their relation to their inheritance from the past." For knowledge, he said, is the "reminiscence by the individual of the experience of the race."

While at Trinity College, Cambridge, which he entered as a student in 1880, Whitehead attended lectures only in mathematics. But he also began to develop an interest in philosophy which was stimulated through nightly conversations, Platonic in method, between friends, undergraduates and members of the staff. These conversations started at dinner and lasted until about ten in the evening, and were followed in his case by two or three hours' work in mathematics. They covered a range of subjects—politics, general philosophy, religion, and literature—and occasioned miscellaneous reading outside of mathematics. At one time, Whitehead says in his reminiscences, he nearly knew by heart parts of Kant's *Critique of Pure Reason*. Hegel, however, he was never able to read, because he initiated his attempts by studying some remarks of Hegel's on mathematics, remarks which seemed to him complete nonsense.

The "Apostles," a club which had been formed by Tennyson and some friends in the 1820's, was another center of discussion. The eight or ten undergraduate members with older members who often attended, met on Saturday nights from ten until the early hours of the morning. Henry Sidgwick, the philosopher, was one of the men who met and talked with this group, and judges, scientists, and members of Parliament, then well known, occasionally came up from London and joined the discussions. This, Whitehead records, was a wonderful influence.

Whitehead's connection with Cambridge was unbroken from 1880 until 1910 when he left to go to London. He became a fellow of Trinity College in 1885, and later was appointed Lecturer, and then Senior Lecturer in mathematics. Two important influences on his life and work during those years at Cambridge might be mentioned here. One was his marriage in 1890. His wife, he says, had a fundamental effect on his understanding of life and the world. In his words "her vivid life has taught me that beauty, moral and aesthetic, is the aim of existence; and that kindness, love, and artistic satisfaction are among its modes of attainment. Logic and Science are the disclosure of relevant

patterns, and also procure the avoidance of irrelevancies."[1] The second influence was Bertrand Russell who was his student in the 1890's, and later his collaborator on the *Principia Mathematica*, their great work in mathematical logic.

In London Whitehead was very active in educational affairs. He was Dean of the Faculty of Science at the University of London, and a chairman or member of several committees which were concerned with education at various institutions. He served successively as lecturer on applied mathematics and mechanics and reader in geometry at the University College, and then as Professor of Applied Mathematics at the Imperial College of Science and Technology from 1914 to 1924.

His fourteen years in London changed Whitehead's views of the problems of modern education, he says, the problems introduced by "the seething mass of artisans seeking intellectual enlightenment, of young people from every social grade craving for adequate knowledge."[2] The University of London had been recently remodeled by Lord Haldane, and was a confederation of various institutions of different types striving to meet this problem of modern life. Whitehead remarked later that the success of the Cambridge method of teaching when he was a student and lecturer there was "a happy accident dependent on social circumstances which have passed away—fortunately. The Platonic education was very limited in its application to life."[3] Through the rest of his life Whitehead continued to strive against the tendency of universities to exclude themselves from the activities of the world around them, against departmentalization, and against the conventionalization of knowledge.

While in London, in addition to his participation in the Royal Society, Whitehead belonged to the London Aristotelian Society, which, he says, was a pleasant center of discussion, and the papers he presented to the Society bear this out, combining physics, mathematics, and philosophy in a rich mixture. His philosophical importance gradually increased during his later years in London, becoming so great that at the age of sixty-three, when he was ready to retire from active university life, he was invited to become a Professor of Philosophy at Harvard University. He accepted and held this position until 1936, when he became Professor Emeritus.

In Cambridge, Massachusetts, the Whiteheads lived in a modest apartment, to which his students came for coffee and conversation during his years as a teacher there. These informal talks were as much a part

[1] Autobiographical Notes, from the *Philosophy of Alfred North Whitehead*, v. 3 of the *Library of Living Philosophers*, Northwestern University, 1944, p. 8.

[2] *Ibid*, p. 12.

[3] *Ibid*, p. 8.

of his influence on students and friends as were his lectures, conveying as they did to his listeners a sense of the exciting adventure of ideas. It was his belief that ideas *are* an adventure, and that the philosopher must not be a mere scholar or critic, but must be speculative and daring. He identifies life with this experience and sense of disclosure, and this emphasis on imaginative novelty.

Whitehead's intellectual life, we find, divides naturally into two parts —his life until he was almost sixty, during which he was concerned chiefly with mathematics, though with a sustained interest in the problems of philosophy—and his life from that time until his death a few months ago, during which he devoted himself almost entirely to philosophical speculation. Most of this latter period of his activity occurred while he lived in America, and amounted thus almost to a second life, in location as well as in activity and purpose.

In the first period Whitehead's mathematical researches were interestingly varied, the most important probably being those in geometry and in the reduction of mathematics to logic. His first published book was *A Treatise on Universal Algebra*, which he began in 1891 when he was thirty years old, and which was published in 1898. He says of it

"The ideas in it were largely founded on Hermann Grassmann's two books, the *Ausdehnungslehre* of 1844, and the *Ausdehnungslehre* of 1862. The earlier of the two books is by far the most fundamental. Unfortunately when it was published no one understood it; he was a century ahead of his time. Also Sir William Rowan Hamilton's *Quaternions* of 1853, and a preliminary paper in 1844, and Boole's *Symbolic Logic* of 1859, were almost equally influential on my thoughts. My whole subsequent work on Mathematical Logic is derived from these sources." (*Ibid*, p. 9.)

Whitehead discovered that his projected second volume of *Universal Algebra* was practically identical with Bertrand Russell's projected second volume of *The Principles of Mathematics* (1903) and so, Whitehead says, "we coalesced to produce a joint work." Before this joint effort resulted in the great *Principia Mathematica*, however, Whitehead published two smaller tracts on *The Axioms of Projective Geometry* (1906) and *The Axioms of Descriptive Geometry* (1907) in the series The Cambridge Tracts in Mathematics and Mathematical Physics. *Principia Mathematica* then came out after eight years of work in three large volumes published from 1910 to 1913, and in a second edition from 1925 to 1927. In this tremendous work, the authors endeavored to show that all of the basic ideas and operations of arithmetic (and hence of all mathematics) could be derived from premises and ideas which are purely logical in their nature. While they did not claim that their premises are necessary for this undertaking, they did claim that they are sufficient for the purpose. *Principia Mathematica* was certainly

one of the most influential works in the development of modern symbolic logic. And with its publication Whitehead began to turn his attention (at least so far as his published works are concerned) more and more to problems of natural philosophy which are not purely mathematical in their character.

The titles of some of his important papers and books published while he was in London show his increasing interest in philosophical problems: *La Theorie Relationniste de l'Espace* (1916), *An Enquiry Concerning the Principles of Natural Knowledge* (1919), *The Concept of Nature* (1920), and similar titles. His concern in most of these works is to construct a logical and self-sufficient framework of concepts which would describe nature in its most general terms, and which would also describe space and time, or space-time, in sufficient detail, while starting from very general principles, to provide the logical basis for measurement, and hence for mathematical science. His main intention is to give a classificatory analysis of perceived experience, exhibiting just what he finds in it as a datum, excluding any consideration of a mind or knower. He treats nature, or experience, as a self-contained entity to which considerations about the nature of mind are irrelevant. It is important to specify this last point, because as his philosophical ideas developed, he became more and more interested in this relation of a percipient mind to the world of events which it perceives, and the first stage of his philosophical development drew to a close.

Before proceeding to a consideration of his later philosophical works, it is interesting to notice the publication of his book *The Principle of Relativity*, in which Whitehead presents his own alternative to Einstein's general gravitational equations, based on his objections to a curved space which was not homogeneous. He felt that the structure of space could not depend on the distribution of masses in it, any more than it could depend on the color of the objects in it. He was never able to accept Einstein's empirical approach to the problem of the curvature of space, and indeed the problem of measurement in general, and the author has been told that Einstein is equally unable to understand Whitehead. Whitehead's basic objection seems to be that measurement of *space*, as contrasted with the description of the vagarious behavior of material bodies, presupposes a specification of properties of parallelism and congruence, i.e., the structure of space itself. It follows, he believed, that Einstein's general theory of relativity, which makes measurement basic, is untenable.

It may be remarked in this connection that while Whitehead is certainly correct in saying that there are difficulties involved in Einstein's procedure,[4] in practice they do not seem insurmountable, and the valid-

[4] Cf., for example, G. C. McVittie, *Cosmological Theory*, Methuen, London, p. 42.

ity of Whitehead's criticism can be disputed, provided the question of whether relativity deals with actual space is left mooted. However, the brevity of this biography does not allow further discussion of this point here.

Soon after Whitehead accepted a Professorship of Philosophy at Harvard University, the first book of his new and more mature philosophical thought appeared, *Science and the Modern World* (1926), and then in quick succession *Religion in the Making* (1926), *Symbolism* (1928), and the great Gifford Lectures, *Process and Reality* (1929), his *magnum opus* in the philosophical field.

In *Science and the Modern World* and *Religion in the Making* Whitehead brings the aesthetic, religious, and moral aspects of experience into his purview, and also increases the emphasis on the creative character of nature. In accordance with this emphasis, Whitehead shows a tendency to merge the act of experience with the act of self-creation. And this act of self-creation, or self-realization, cannot be fully described without reference to its realization of value, purpose, and aesthetic enjoyment, these latter terms all being essentially synonymous. This emphasis on the creative act of self-realization, which is a realization of aesthetic value, quickly comes to dominate Whitehead's philosophical writing, culminating in his great cosmological symphony, *Process and Reality*. The intuitions of poets, religious leaders, and artists now are fully as important to a full understanding of the cosmos as are the insights of scientists and mathematicians. So not only Euclid and Riemann, and Einstein and de Broglie, but also Wordsworth and Plato and Jesus and the great Hebraic prophets must be consulted if one is to understand the world. And one is led from a realization of the omnipresence of value in the world to a realization of the character of God, who envisages and desires an ideal situation for the world, and thus provides a standard of value to be reckoned with, whether or not it is conformed with. Thus each epoch of history has its own ideal values, to be striven for and realized as much as possible, if the divine will is to be obeyed. Whitehead's persuasion that there is an immanent and universal scheme of values in the world becomes one of the fundamental premises of experience in his cosmology, more fundamental than the hypothetical revelations of science.

Although Whitehead published several books after *Process and Reality*, he introduced no important changes in his doctrine in them, spending the remaining twenty years of his life in amplifying and commenting on the consequences of his premises. Thus his period of creative philosophical activity falls almost entirely between the fifty-fifth and seventieth years of his life, being most intensely concentrated

in the five years from 1923 to 1928. So in *Adventures of Ideas* (1933) and *Modes of Thought* (1938) we find mainly more popular restatements and amplifications of concepts already expressed in *Process and Reality*.

It is, of course, impossible to give an adequate summary of *Process and Reality* in the space of a few pages, and difficult to convey even a sense of the direction and purpose of the book. Whitehead calls it an "essay in cosmology," and it is that in a very large sense—not the cosmology of relativistic physics, or of E. A. Milne, but a description of the basic factors involved in the structure and behavior of the universe, and in the "actual occasions" (events) that are its building blocks. His cosmos is a logical structure embodying his insights into the creative act of experience, the struggle for the realization of value, the nature of causation and perception, the geometrical structure of space-time (and the proper order of its elucidation) and the role of God in this cosmos. In attempting to present this unified picture of the world, he finds that various special disciplines for the discovery of fact and order in nature are all, in themselves, incomplete. For example, he says in *Modes of Thought*[5] (a very readable small work):

> "The terms 'morality,' 'logic,' 'religion,' 'art,' have each of them been claimed as exhausting the whole meaning of importance.... There are perspectives of the universe to which morality is irrelevant, to which logic is irrelevant, to which religion is irrelevant, to which art is irrelevant.... No one of these specializations exhausts the final unity of purpose in the world."

While regarding all disciplines as thus limited in their importance, Whitehead continued to the end of his writings to emphasize the central importance of mathematics. He says in his essay on *Mathematics and the Good*.[6]

> "Mathematics is the most powerful technique for the understanding of pattern, and for the analysis of the relationships of patterns.... If civilization continues to advance, in the next two thousand years the overwhelming novelty in human thought will be the dominance of mathematical understanding. The essence of this generalized mathematics is the study of the most observable examples of the relevant patterns; and applied mathematics is the transference of this study to other examples of the realization of these patterns."

[5] From A. N. Whitehead: *Modes of Thought*, p. 16 (copyright 1938 by the Macmillan Company, and used with their permission).
[6] *The Philosophy of Alfred North Whitehead*, *loc. cit.* p. 678.

3.

FOREWORD

In the period between the two great world wars there arose in Poland a group of brilliant mathematicians whose interest leaned largely, though by no means exclusively, toward mathematical logic. Included in this so-called Polish School of Mathematicians, among others, are the names of L. Chwistek (symbolic logic), J. Lukasiewicz (many-valued logics), S. Banach (vector spaces), L. Infeld (relativity theory), K. Kuratowski (topology), and W. Sierpiński (theory of real functions).

One of the deepest tragedies in the history of human thought occurred when this group of illustrious mathematicians was all but completely wiped out in the Nazi onslaught during World War II in which virtually all the 3,100,000 Jews in Poland were exterminated. In Warsaw alone, in 1944, the Nazis killed 40,000 Jews, expelled the few remaining survivors, and then deliberately demolished the city. As E. T. Bell has said: "The great school of Polish mathematicians followed the Vienna Circle into death or exile. What had taken twenty years to gather was dispersed and in part obliterated in about twenty days."

Fortunately, some of these Polish mathematicians survived; some fled to America. It is the enduring faith of such people that explains the miraculous rebirth of the present Polish Academy of Sciences, established in Warsaw in 1952, as well as the Institute of Mathematics, founded in Warsaw in 1948, with branches in Cracow and Wroclaw.

Waclaw Sierpiński–[1] Mathematician

Matthew M. Fryde

In 1962 scholars throughout the world celebrated the eightieth birthday anniversary of Waclaw Sierpiński, one of the greatest mathematicians of our time, the head of the Warsaw School of Mathematics and author of numerous works of lasting value.

Prior to the twentieth century. Poland was represented on the mathematical Olympus only by Hoene-Wronski. It is the Warsaw School of Mathematics under the leadership of Waclaw Sierpiński, together with the Lwów School of Mathematics under Hugo Steinhaus and Stefan Banach, which secured for the Polish mathematicians a foremost place in modern mathematics.

Waclaw Sierpiński was born in Warsaw on March 14, 1882. His father, a physician, looked after his education which, from the very beginning, was thorough and solid. The young Sierpiński attended both high school and university—at that time Russian—in his native city. His unusual mathematical talent attracted the attention of his teachers, and especially of G. F. Voronoy, a noted mathematician and professor at Warsaw University. Under his influence Sierpiński specialized in the Theory of Numbers, of which C. F. Gauss once said that it is the Queen of Mathematics. The first works of Sierpiński were written in the service of that Queen. For one of them he was awarded by Warsaw University, as early as 1904, a gold medal. Sierpiński's contributions to the Theory of Numbers are numerous. They consist of long series of important papers and monographs. In 1914 Professor Sierpiński published in Warsaw a treatise, *Teoria liczb* (Theory of Numbers), reprinted with some changes in 1925 and thoroughly revised in 1950. An additional second volume was published in 1959. It contains new results obtained since 1950 as well as new materials. These two solid volumes, condensed into one, are to appear soon in an English translation. While working later in many other important fields of mathematics, Professor Sier-

[1] Reprinted from *The Polish Review*. New York, VIII (1963), 1–8.

piński never lost deep interest in the Theory of Numbers to which he constantly contributes by new papers and monographs. Professor Sierpiński possesses a very rare gift to write not only for accomplished scholars, but also for beginners, always with the same lucidity. Some of his special monographs in the field of the Theory of Numbers are models of presentation, where the originality and novelty of thought are combined with exemplary clarity. Among these works the following, published recently, should be mentioned:

Trójkaty Pitagorejskie (Pythagorean Triangles), Warsaw, 1954. A translation into Russian appeared in 1959 and an English version by Dr. A. Sharma, in 1962 (The Scripta Mathematica Studies, No. 9, New York, 1962); *O rozwiazywaniu równan w liczbach calkowitych* (Solutions of Equations in Integers), Warsaw, 1956; *O rozkladach liczb wymiernych na ulamki proste* (Partitions of Rational Numbers into Unit Fractions), Warsaw, 1957; *Czym sie zajmuje teoria lizab* (The Subject of the Theory of Numbers), Warsaw, 1957; *Co wiemy i czego nie wiemy o liczbach pierwszych* (What We Know and What We Do Not Know of Prime Numbers), Warsaw, 1961. Professor Sierpiński also completed (in collaboration with Dr. Jerzy Los) a textbook of Theoretical Arithmetics (in Polish), the second edition of which appeared in Warsaw in 1959, and a textbook of Higher Algebra (in Polish), with a special chapter on the Galois Theory by Professor A. Mostowski, 2nd ed., Warsaw, 1951.

After having finished, in 1904, his university studies, Sierpiński started a teaching career in a Russian high school in Warsaw. He resigned in connection with a mass protest of the Poles, in particular of Polish intellectuals, students and school children, against the obligatory use of the Russian language at Warsaw University and in the schools attended by Polish youth. To manifest his personal protest, Sierpiński withdrew from Warsaw University the manuscript of his work, for which he had been awarded the prize, already mentioned.

In the fall of 1905 Sierpiński entered the Jagellonian University in Cracow, from which he graduated, in 1906, with a Ph.D. degree. He returned to Warsaw, where he taught in Polish high schools set up following the mass boycott of the Russian schools by the Poles. He also joined as lecturer the newly established Polish center for higher learning, which became the nucleus of the future Free Polish University in Warsaw. In the summer of 1908 he became a member of the Warsaw Scientific Society.

Sierpiński's works, published during the early period of his scholarly activity (1906–1910) are mostly devoted to the Theory of Numbers. Soon, however, another highly important branch of mathematics came into the focus of his interest: the Theory of Sets.

Created by Georg Cantor, this theory was for a long time grossly neglected or bluntly rejected by the older generation of mathematicians, and in any case was not taught at the universities. To what extent the Theory of Sets was unknown in the first decade of the twentieth century even to highly educated mathematicians—and young Sierpiński was certainly one of them—is fully and spectacularly attested by the fact that Sierpiński, destined to play such an important role in the future development of the new discipline, did not know, in 1907, that it existed. Gradually he became immersed in the new world of concepts and theories and his first publication (in Polish) in the field of the Theory of Sets appeared as early as 1908.

In 1908 Professor J. Puzyna of the University of Lwów (which was a Polish university) invited Waclaw Sierpiński to join the faculty of mathematics. Among the courses given by the young professor was, since 1909, the Theory of Sets. The same year a mimeographed edition of his lectures appeared and a textbook—*An Outline of the Set Theory* (in Polish)—was completed and published in Warsaw, in 1912. It introduced the new discipline to generations of Polish mathematicians and its role in this respect can never be sufficiently emphasized. It contributed immensely to the spectacular development of mathematics in Poland and became a classic. The second edition was published in 1923 and an enlarged edition, divided into two parts, followed in 1928, the first part being devoted to the *Theory of Sets* and the second to *General Topology*. Some sections of this work appeared in English as *General Topology*, Toronto, 1934 and 2nd ed., 1952. Certain branches of the Theory of Sets were treated in special monographs: *Leçons sur les nombres transfinis*, Paris, 1928, 2nd ed., 1950, and *Cardinal and Ordinal Numbers*, Warsaw, 1958. For the Algebra of Sets, a monograph *Algèbre des ensembles* was published in Warsaw, in 1951. Professor Sierpiński devoted also a special monograph to the discussion of the hypothesis of the continuum: *Hypothèse du continu*, 1st ed., Warsaw, 1934, 2nd ed., New York, 1956, which belongs, as it was stressed by him, to the most difficult problems of contemporary mathematics. In this monograph he does not intend to solve the problem, his purpose being only to explain the consequences of the hypothesis of the continuum. In 1962 a comprehensive *Introduction to the Theory of Sets and Topology*, translated from Professor Sierpiński's French manuscript by Professor A. Sharma, appeared in India. The Polish original of this short *Introduction* was published in Lwów in 1930, 2nd ed., Warsaw, 1947. To the Theory of Sets Professor Sierpiński devoted numerous papers, published in various periodicals and bulletins of academies of sciences. The complete list of these publications would require a special biblio-

graphical essay. It is sufficient to note that in the bibliography to his monograph *Cardinal and Ordinal Numbers* Professor Sierpiński refers to no less than seventy-one of his own publications.

The important role of the Theory of Sets in moulding and shaping the whole structure of modern mathematics cannot be discussed in the present article. In view of the fact, however, of Professor Sierpiński's paramount role in the development of that theory, a few historical remarks seem necessary at this point.

Ridiculed at first, the Theory of Sets became gradually the backbone of modern mathematics, which is now absolutely unthinkable without this new basis. It was rightly said by one of the authorities in this field that the Theory of Sets is "one of the greatest creations of the human mind. In no other science is such bold formation of concepts found, and only the Theory of Numbers, perhaps, contains methods of proof of comparable beauty" (E. Kamke, *Theory of Sets*, New York, 1950, p. VII). Under the impact of the Theory of Sets new disciplines came into being, while other branches of mathematical science were developed to a very high standard, among them Topology and the Theory of Functions of a Real Variable. As a matter of fact, every branch of mathematics was affected by the new discipline and some of them were revolutionized by it. Besides, the theoretical foundations of mathematics became thoroughly transformed and modern logic has undergone a spectacular growth. Professor Kazimierz Kuratowski, one of Sierpiński's most eminent pupils, recently pointed out that, owing to the fact that the development of the Theory of Sets went at first in a purely abstract direction and because the methods of the new discipline were strange as compared to those commonly used, it was only when it became apparent how useful the Theory of Sets was for other branches of mathematics that the new discipline became accepted without reservations (Kazimierz Kuratowski, *Introduction to Set Theory and Topology*, New York, 1961, pp. 19–20).

The great importance of the Theory of Sets was by no means clear in 1909. The fact that Waclaw Sierpiński became already at that time a consistent and eloquent exponent of the new discipline attests *per se* to his vision and imagination, to the great power of his intellect, and to his deep penetration into the concepts and structure of the new-born branch of mathematics which was destined to inaugurate a new era.

Already prior to the outbreak of the First World War, Sierpiński's reputation as an outstanding Polish mathematician was firmly established. In 1911 and 1913 prizes were awarded to him by the Academy of Arts and Sciences in Cracow; in 1917 he was appointed corresponding member and received another prize; in 1921 Waclaw Sierpiński became an active member of that Academy. From 1931 Professor Sierpiński was

President of the Warsaw Scientific Society, and in 1951 a medal was struck to commemorate twenty years of his presidency.

In 1907 Sierpiński met in Warsaw Zygmunt Janiszewski, a student of mathematics at Paris University. After this outstanding mathematician obtained in 1911 his Ph.D. degree from Paris University on the basis of a thesis, suggested by Lebesgue, in the field of Topology, Sierpiński secured for the young scholar a teaching position at the University of Lwów. Through Janiszewski Professor Sierpiński made the acquaintance of another highly gifted young mathematician, Stefan Mazurkiewicz. On Sierpiński's initiative Mazurkiewicz came to Lwów, where he obtained his Ph.D. degree on the basis of the solution of a problem in the Theory of Sets, proposed by Sierpiński. The collaboration with these two young mathematicians, interrupted in 1914 by the outbreak of the First World War, played later an important role in Sierpiński's work in the organization of mathematical research and teaching in Poland.

During the early period of the war, Sierpiński was forced to live in the Russian town of Viatka. Then he was able to move to Moscow, where he met the eminent Russian mathematician N. N. Lusin, a specialist in the field of the Theory of Functions of a Real Variable. The contact with Lusin was fruitful in many respects. The Theory of Functions of a Real Variable, created by French mathematicians at the end of the nineteenth century, was successfully developed by Sierpiński, partly in close collaboration with Lusin. A series of important studies was published jointly by them. The details concerning Professor Sierpiński's role in the above mentioned branch of mathematics (as well as in other fields) are to be found in a lucid article by Professor Edward Marczewski "O dzialalnosci naukowej Waclawa Sierpinskiego" (The Scholarly Activities of Waclaw Sierpiński) in the publication *VI Zjazd Matematyczny. Jubileusz 40-lecia dzialalnosci na katedrze uniwersyteckiej profesora Waclawa Sierpinskiego. Warszawa, 23.9.1948* (The Sixth [Polish] Mathematical Congress. The 40th Anniversary of the University Teaching of Professor Waclaw Sierpiński. Warsaw, September 23, 1948), issued by the Jubilee Committee, Warsaw, 1949.

In February, 1918, Sierpiński returned from Russia to Lwów to resume his teaching that had been interrupted by the war. In the fall of the same year he received an invitation from Warsaw University, at that time already Polish. There he met Janiszewski and Mazurkiewicz. In his article "The Warsaw School of Mathematics and the Present State of Mathematics in Poland" (*The Polish Review*, Vol. IV, No. 1–2, 1959) Professor Sierpiński tells how the epoch-making decision to create in Warsaw a special periodical, the world-famous *Fundamenta Mathematicae*, was soon reached by him and his two younger friends:

In 1919, three of us met as the first professors of mathematics in the reborn Polish University at Warsaw. There we decided to found a periodical dedicated to the theory of sets, topology, the theory of function of a real variable and mathematical logic, which would publish studies in French, English, German, and Italian. It was thus that *Fundamenta Mathematicae* was born.

The first copy of the new periodical appeared in 1920. Soon *Fundamenta Mathematicae* became the leading journal in the branches of mathematics to which it was dedicated. Among many eminent foreign contributors the names of Borel, Lebesgue, Hausdorff, Lusin, Denjoy, Alexandroff can be mentioned. The thirty-two volumes of *Fundamenta Mathematciae,* published during the period 1920–1939, contain 972 studies contributed by 216 authors, among them sixty-two by Polish mathematicians. Waclaw Sierpiński himself published there 201 studies, of which fifteen jointly with other mathematicians. As a matter of fact, the modern history of the Theory of Sets and of the Theory of Functions of a Real Variable are to a great extent embodied in the volumes of *Fundamenta Mathematicae*. The unusual specialization and concentration, well planned since the very beginning, determined the spectacular success of *Fundamenta Mathematicae*. The new periodical, being the center of a mighty gravitational field which attracted specialists from all over the world, became at the same time the organ of a new school: the Warsaw School of Mathematics.

Janiszewski died before the first volume of *Fundamenta Mathematicae* was published, thus Sierpiński and Mazurkiewicz were the editors. When the latter died in 1945, Kuratowski took his place as co-editor. *Fundamenta Mathematicae* is still the center of mathematical research with Professor Sierpiński as its honorary editor.

It is impossible to describe, within the compass of the present short article, all the studies published by Professor Sierpiński—their number exceeds 700. It is equally impossible to give an adequate picture of his tremendous activity as the great organizer of mathematical research and learning in Poland. Yet it is imperative to stress that this great creative mathematician is also a great and successful teacher. Showing a strong aversion to any form of splendid isolation, sometimes characteristic of great men of science, Professor Sierpiński is not only immersed in his own research, but he lives in a kind of "ideal space," created by him, together with his younger colleagues, former pupils and students. In spite of strenuous brain work, constantly performed and apparently absorbing him entirely, Sierpiński always finds time for anybody seeking his guidance, help or encouragement. This strongly marked trait of his character should never be forgotten when one attempts to evaluate the success and glory of the Warsaw School of Mathematics. No wonder that around Sierpiński a circle of men of great talent was formed.

Among the noted mathematicians (or logicians) belonging to that School the following should be mentioned: Aronszajn, Borsuk, Eilenberg, Knaster, Kuratowski, Marczewski, Mostowski, Splawa-Neyman, Saks, Tarski, Walfisz, Zygmund and others. Sierpiński's pupils became teachers, and their pupils and pupils' pupils all continue the great tradition of scholarship.

In May, 1939, the University of Paris awarded Waclaw Sierpiński a doctor's degree *honoris causa*, and November 9, 1939 was the date fixed for the official ceremony. The outbreak of the war made Sierpiński's journey to Paris impossible, yet the University conferred the honorary degree *in absentia*. The formal presentation took place in 1960.

The inhuman ruthless oppression of the Polish population by the German authorities during the Second World War paralyzed normal scholarly activities in Poland. Professor Sierpiński, however, who was employed nominally in the book-keeping department of the City Hall in Warsaw, managed even under these circumstances to continue his creative work. Some results obtained were smuggled out to Rome and appeared in *Acta Pontificiae Academiae Scientiarum*. In his apartment mathematics was clandestinely taught and twice a month Professor and Mrs. Sierpiński entertained small groups of friends. The Warsaw insurrection of 1944 and the following barbarous destruction of the city by the Germans put an end to all efforts of this courageous and indefatigable man. Part of the building, where Sierpiński's apartment was located, escaped destruction, yet in October 1944, a special German destruction squad—the *Brandkommando*—set the remaining part of that building on fire. A valuable library, containing not only a rich collection of books but also many manuscripts and documents, perished in flames. Among the documents destroyed were letters of famous mathematicians such as Cantor, Vitali, Vivanti, Lebesgue, Schoenfliess, Zermelo, and many others. Sierpiński and his wife were deported by the Germans to a locality in the neighborhood of Raclawice, where they remained until February, 1945. From there Sierpiński, then a man of over sixty, walked almost forty miles to Cracow, to start teaching. In the fall of the same year he returned to Warsaw—at that time in ruins—and with his indomitable energy started again to write and to teach.

On the occasion of Professor Sierpiński's jubilee, in 1948, many institutions and eminent scholars sent their congratulations. Some excerpts of these messages are quoted below.

The American Mathematical Society wrote:

... The tremendous progress made by the Polish School of Mathematics in the period immediately following the First World War was, in large measure, inspired

by Professor Sierpiński. By his own research and by the inspiration of the brilliant students, he brought Polish Mathematics to the very forefront in our science....

In the message of the London Mathematical Society we read:

...British mathematicians, in common with their colleagues throughout the world, have long admired your outstanding contributions to mathematics, contributions which have advanced and enriched our subject and which will occupy a permanent place in the literature. In particular, your work on the continuum hypothesis, general topology, and set theory, will long be a source of inspiration to mathematicians of all countries....

The French Academy of Sciences in Paris declared:

...Vos recherches sur la théorie des ensembles et celle des fonctions de variables réelles, votre action sur la brillante Ecole de mathématiciens qui s'est formée en Pologne, sous votre impulsion, de 1919 à 1939, la haute qualité du recueil des *Fundamenta Mathematicae* fondé et dirigé par vous, vous ont donné une place éminente parmi les mathématiciens de notre temps....

Professor Sierpiński is *doctor honoris causa* of many famous universities, among them of Paris, Lwów, Amsterdam, Bordeaux, and Prague, member of various learned societies and academies of sciences, among them the Polish Academy of Science. Since 1947 he has been a foreign member of the Academia Nazionale dei Lincei in Rome. In June 1948 he was elected a corresponding member of l'Institut de France and since December 1960 as *Associé Etranger* of the same body. Sierpiński is also honorary member of numerous mathematical societies. It would require many pages to give a full list of the honors and awards bestowed upon this great scholar.

In 1962 the American Mathematical Society invited Professor Sierpiński to give a series of lectures in various American universities. He accepted this invitation and with almost youthful vigor traveled throughout the United States. Back in his native Warsaw, Professor Sierpiński continues his work, only to be interrupted from time to time by visits to foreign universities and participation in congresses of mathematicians.

FOREWORD

One of the most brilliant mathematicians of the first half of the twentieth century, John von Neumann is noted for outstanding achievements in several more or less unrelated areas: quantum mechanics, game theory, electronic computers, logic, and functional analysis.

In the first area he showed that Schrödinger's wave mechanics and Heisenberg's matrix mechanics were mathematically equivalent. In the case of game theory, he virtually created the branch of mathematics known today as decision-making, strategy and the theory of games. Von Neumann was equally at home in pure mathematics and applied mathematics. He made significant contributions to the design and construction of early electronic computers such as were used in producing the H-bomb.

The article that follows illuminates both the man and his work.

Scientific Work of J. von Neumann

Herman H. Goldstine
and
Eugene P. Wigner

Even before the present age of specialization, few people have ever contributed significantly to several branches of science, and all of them have a permanent record in the annals of the history of science. John von Neumann made fundamental contributions to mathematics, physics, and economics. Furthermore, his contributions are not disjoined and separate remarks in these fields but arise from a common point of view. Mathematics was always closest to his heart, and it is the science to which he contributed most fundamentally.

John von Neumann was born in Budapest on 28 December 1903. He studied in Berlin, Zürich, and Budapest, receiving his doctor's degree in 1926. After serving as Privatdozent in Berlin and Hamburg, he was invited to Princeton University in 1930. Following 3 years there, he became professor of mathematics at the Institute for Advanced Study, a position which he held for the rest of his life. In 1955 he was appointed to the U.S. Atomic Energy Commission and served brilliantly in this post until his death on 8 February 1957.

The earliest significant mathematical work of von Neumann concerns mathematical logic, in which he was a forerunner of the epochal work of Gödel. His accomplishments can be summarized under two headings: axiomatics of set theory and Hilbert's proof theory. In both of these subjects he obtained results of cardinal importance.

Von Neumann was the first to set up an axiomatic system of set theory satisfying the following two conditions: (i) it allows the development of the theory of the *whole* series of cardinal numbers; (ii) its axioms are finite in number and expressible in the lower calculus of functions. Moreover, in deriving the theorems on sets from his axioms, he gave the first satisfactory formulation and derivation of definition by transfinite induction. Von Neumann's work on this subject already showed his power and the elegance of much of his later work. It contained a full clarification of the significance of the axioms with regard to the elimination of the paradoxes. He first showed how the paradoxes are related to the theory of types and then proved that a set exists (this implies that it does not lead to contradictions) if, and only if, the multitude of its elements is not of the same cardinality as the multitude of all things. He also demonstrated that this proposition implies the axiom of choice.

With regard to Hilbert's proof theory, von Neumann clarified the concept of a formal system to a considerable extent. His articles contain the first unobjectionable proof for the fundamental theory that the classical calculus of propositions and quantifiers as applied to computable functions and predicates is consistent.

The work of von Neumann which will be remembered longest concerns the theory of the Hilbert space and of operators in that space. His papers on this subject can be divided into three groups: (i) the properties and structure of Hilbert spaces as such; (ii) studies of linear operators involving in essence only a single operator; (iii) studies of whole algebras of operators.

Von Neumann gave the first axiomatic treatment of Hilbert space and described the relation of Hilbert spaces to all other Banach spaces. A good exposition of his point of view on linear spaces is given in his book on functional operators.

In a remarkable paper, von Neumann gave the complete theory of extensions of Hermitian operators H on Hilbert space to maximal and self-adjoint operators, by means of the Cayley transform $(H+iI)(H-iI)^{-1}$. By the same transform, he established the spectral theorem for self-adjoint operators; that is, he constructed a set of projection operators $E(\lambda)$ with the property that H (where $H = H^{*}$) admits a spectral resolution $(Hf, g) = \int \lambda d(E(\lambda)f, g)$. He derived a similar theorem for normal operators. The spectral theorem has enormous importance in applications, and von Neumann's work has been of great influence.

Partly in collaboration with Murray, von Neumann founded the theory of weakly closed, self-adjoint algebras ("rings") of bounded linear operators. They first studied "factors"—that is, rings generalizing simple algebras—and developed a "direct sum" theory for rings of operators. The effect of von Neumann's work here is enormous. A whole school has grown up in the past decade devoted to a study of operator rings and their abstract analogs.

In pursuing his researches on rings of operators he was led to introduce the notion of a dimension function into ring theory and found thereby "geometries without points." He developed this theory into his important continuous geometry, which was the subject of his 1937 colloquium lectures to the American Mathematical Society.

The influence of von Neumann's interest in groups can be detected in all phases of his work on operators. In particular, the direct sum theory has many applications in the theory of unitary representations of non-Abelian noncompact groups, as is shown in the work of Mackey, Godemont, Mautner, Segal, and Gel'fand and his school. Von Neumann's work on unbounded operators has heavily colored analysis in

the past 25 years. It seems safe to predict that his work on operator rings will color it even more strongly during the next 25.

His contribution to the theory of groups did not stop here. He was the first to show that every subgroup of a matrix group is a Lie group. This result is fundamental to the present techniques for analyzing locally compact groups. He also showed that every compact group can be approximated by Lie groups, and as a consequence that every compact locally Euclidean group is a Lie group. His work on almost periodic functions on groups won for him the Bôcher prize.

His elegant proof of the ergodic theorem stands as one of his important results. Its ramifications have had a profound influence on the study of dynamical systems.

Von Neumann was one of the founders of the theory of games. In spite of the nearly 30 years that have passed since von Neumann's first paper was written on this subject, and in spite of the intensive development of the theory in these 30 years, there is very little in his first paper that would be revised today. It is, as are many of his early papers, strongly under the influence of the axiomatic thinking and gives a formal system which permits a complete description of all the intricacies of a game, with play and counterplay, chance and deception. The paper contains a rigorous definition for the concepts of pure strategy (a complete plan, formulated prior to the contest, making all necessary decisions in advance) and of mixed strategy (the use of a chance device to pick the strategy for each contest). Although similar concepts were used before (by Zermelo and by Borel), no one had used them before with the same incisiveness as von Neumann did when he established the "minimax theorem" for zero sum two-person games. This theorem, which proved valuable for von Neumann's studies in economics, also gave the key to the analysis of games with more than two players, permitting the formation of alliances and camps between the players.

The book, *The Theory of Games and Economic Behavior*, by him and Morgenstern, has affected decisively the entire subject of operations research. Indeed, it may well be said that the present-day importance of the subject results from the influence of this monumental work.

The preceding three subjects are the ones which come to mind at once when writing about von Neumann's contributions to mathematics. However, they are surely not the only fields which have profited from his fertile imagination. He has made significant contributions literally to every branch of mathematics, with the exception of topology and number theory. His knowledge of mathematics was almost encyclopedic —again excepting the afore-mentioned two fields—and he gave help and advice on many subjects to collaborators and casual visitors, possibly to

a greater extent than any other present-day mathematician.

It would be very difficult to tell which of von Neumann's contributions to theoretical physics was the more important: the direc or the indirect ones. Four of his direct contributions are known to all physicists. His recognition that vectors in Hilbert space are the proper mathematical concept to describe the states of physical systems in quantum mechanics is unique in the sense that no other person would have realized this fact for many years. Closely related to this observation is his description of quantum mechanics itself. The sketch of his ideas in this connection, presented in chapter VI of his *Mathematische Grundlagen der Quantenmechanik*, still constitutes inspiring reading. Von Neumann's third main contribution is the application of the concept of the mixture of quantum mechanical states—which he invented independently of Landau—to problems of thermodynamics and statistical mechanics. The considerations on irreversibility, in both classical and quantum physics, were his fourth major contribution. These contributions, and some others of a more specialized nature, would have secured him a distinguished position in present-day theoretical physics quite independently of his indirect contributions.

Von Neumann developed several mathematical concepts and theorems which became important for the theoretical physicist; he probably developed them with these applications in mind. In fact, it often seems to the theoretical physicist that the best of von Neumann's mathematical work was motivated by its projected usefulness in some applied science. From the point of view of the theoretical physicist, his two most important mathematical contributions were the theory of nonbounded self-adjoint or normal operators in Hilbert space and the decomposition of representations of noncompact groups, carried out in collaboration with Mautner (both of these are described in the preceding section). Many of von Neumann's colleagues think that his late work, centered around the development of fast computing machines, was also motivated by his desire to give a helping hand to his colleagues in mathematics' sister sciences.

No appraisal of von Neumann's contributions to theoretical physics would be complete without a mention of the guidance and help which he so freely gave to his friends and acquaintances, both contemporary and younger than himself. There are well-known theoretical physicists who believe that they have learned more from von Neumann in personal conversations than from any of their colleagues. They value what they have learned from him in the way of mathematical theorems, but they value even more highly what they have learned from him in methods of thinking and ways of mathematical argument.

Von Neumann's contributions to economics were based on his theory of games and also on his model of an expanding economy. The theory of games has relevance in many fields outside of economics; it answers a desire first voiced by Leibnitz but not before fulfilled. It has been stated (by Copeland of Michigan) that his theory may be "one of the major scientific contributions of the first half of the 20th century." The theory rests on von Neumann's minimax theorem, whose significance and depth are only gradually becoming clear. The theory gives a new foundation to economics and bases economic theory on much weaker, far less restrictive assumptions than was the case thus far. The current analogy between economics and mechanics has been replaced by a new one with games of strategy. Entirely new mathematical tools were invented by von Neumann to cope with the new conceptual situations found. This work has given rise to the publication by many authors of several books and several hundred articles. His study of an expanding economy is the first proof that an economic system with a uniform rate of expansion can exist and that the rate of expansion would have to equal the rate of interest. This study has deeply influenced many other scholars and will unquestionably become even more significant now that problems of growth are being so widely investigated by economists.

The principal interest of von Neumann in his later years was in the possibilities and theory of the computing machine. He contributed to the development of computing machines in three ways. First of all, he recognized the importance of computing machines for mathematics, physics, economics, and many problems of industrial and military nature. Second, he translated his realization of the significance of computing machines into active sponsorship of a computer—called JOHNIAC by his affectionate collaborators—which served as a model for several of the most important computers in the United States. Third, he was one of the authors of a series of papers which gave a theory of the logical organization and functioning of a computer which reminds one of the axiomatic formulation of mathematics, a subject to which he devoted so much of his early youth. In these papers is also formulated a quite complete theory of coding and programming for machines. Here is the complete notion of flow-diagrams and the genesis of all modern programming techniques. In one of these papers is given the criteria and desiderata for modern electronic computing machines.